Quiz 16

March 31, 2017

Show all work and circle your final answer.

1. Find a power series representation of $f(x) = \dfrac{x^2}{x+2}$. What is the radius of convergence?

$$\frac{x^2}{x+2} = \frac{x^2}{2\left(\frac{x}{2}+1\right)} = \frac{x^2}{2}\left(\frac{1}{1+\frac{x}{2}}\right)$$

$$= \frac{x^2}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \frac{x^2}{2}\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{1}{2}\right)^n x^n$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n\left(\tfrac{1}{2}\right)^{n+1} x^{n+2}}$$

This is a geometric series with $r = \frac{-x}{2}$, so it converges when $\left|-\frac{x}{2}\right| < 1$, or when $-2 < x < 2$. $\boxed{R = 2}$

2. Find a power series representation for $g(x) = \ln(1-5x)$. (Hint: Consider $\frac{1}{1-5x}$.)

$$\frac{1}{1-5x} = \sum_{n=0}^{\infty}(5x)^n = \sum_{n=0}^{\infty}5^n x^n \leftarrow \text{Note: } R = \tfrac{1}{5}$$

$\int \frac{1}{1-5x}\,dx = -\frac{1}{5}\ln|1-5x| + C$ (by u-sub, $u = 1-5x$), so

$$-\tfrac{1}{5}\ln|1-5x| + C = \int \sum_{n=0}^{\infty}5^n x^n\,dx$$

$$= \sum_{n=0}^{\infty}5^n \frac{x^{n+1}}{n+1}$$

$\underline{x=0}$: $-\frac{1}{5}\ln(1) + C = 0$

$$C = 0$$

So $\ln|1-5x| = -5\sum_{n=0}^{\infty}5^n\frac{x^{n+1}}{n+1}$

$$= \boxed{-\sum_{n=0}^{\infty}\frac{(5x)^{n+1}}{n+1}}$$

Note: $R = \frac{1}{5}$ since the radius of convergence doesn't change when we integrate or take derivatives.

3. Find a power series representation for $h(x) = \dfrac{1}{(1+x)^2}$.

$$\frac{d}{dx}\left(\frac{1}{1+x}\right) = \frac{d}{dx}\left(\left(1+x\right)^{-1}\right)$$
$$= -(1+x)^{-2}$$
$$= \frac{-1}{(1+x)^2}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

So $\dfrac{d}{dx}\left(\dfrac{1}{1+x}\right) = \dfrac{d}{dx}\left(\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n\right)$

$$= \frac{d}{dx}\left(1 - x + x^2 - x^3 + \ldots\right)$$
$$= 0 - 1 + 2x - 3x^2 + \ldots$$
$$= \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$$

equal

since the derivative of x^0 was 0

So $\dfrac{1}{(1+x)^2} = -\displaystyle\sum_{n=1}^{\infty} (-1)^n n x^{n-1}$

$$= \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}}$$

Note: The radius of convergence of $\sum_{n=0}^{\infty} (-1)^n x^n$ is 1,
so the radius of convergence of $\sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$ is also 1.